Presheaf models for concurrency

Final report for an individual project supervised by Giovanni Ciná

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1 Introduction

Nowadays there seems to be an abundance of various models for concurrency. Different models are suited for different tasks, but there is still a need to have a unified approach to concurrency. In [6] the authors describe a general categorical framework for relating various models of concurrency through special kinds of adjunctions – coreflections. With that approach, the operations of process algebra are seen as familiar universal constructions in category theory (like sum, product, pullbacks)

In this report we describe the work originally presented in [2] and [7], which is a natural extension of their categorical approach. In their work, Winskel, Nielsen, and their co-authors develop a way to treat models of concurrency (such as transition systems) and notions of equivalence (such as bisimulation) through presheaves. This is different from the coalegbraic approach to transition systems and bisimulation, which is another popular generalization.

For this report we have aimed to present the theorems and lemmas that were partially omitted, or which proofs were omitted, in the seminal works of Winskel and Nielsen.

The report is organized as follows. In the next section we present the established models for concurrency and describe the categories that they form.

The third section contains material about the so called *open morphisms*, morphisms that satisfy a certain path-lifting property. Then, we describe the use of presheaves, as bundles of paths glued together in a certain way, for the models of concurrency. Finally, in the last section we touch upon some more recent related developments: relational presheaves, that are useful for models with algebraic structure on labels.

2 Models for concurrency

In this chapter we briefly go over main models for concurrency studied in the report. For a broad overview please consult [6].

2.1 Transition systems

Transition systems are, perhaps, the most important model in the study of computation. The concept of a transition system is very concrete, yet it is general enough to model a huge array of types of systems.

Definition 1 (Transition system)**.** *A transition system is a tuple* $(S, i, L, \to)$ *where*

1. S *is a set of states;*
2. $i \in S$ *is an* initial state*;*
3. L *is a set of* labels
4. $\to \subseteq S \times L \times S$ *is a transition relation.*

Instead of writing $(s, a, s') \in \to$ *we often write* $s \xrightarrow{a} s'$*.*

Sometimes we write $T : s \xrightarrow{a} s'$, to emphasize that transition $s \xrightarrow{a} s'$ is a part of the transition system T.

Definition 2. *A (partial) function* $f : S_1 + L_1 \to S_2 + (L_2 \cup \perp)$ *is a morphism of transitions systems* $T_1 = (S_1, i_1, L_1, \to_1)$*,* $T_2 = (S_2, i_2, L_2, \to_2)$ *iff*

1. f *is a function on the states, and a partial function on labels, i.e.* $f(S_1) \subseteq S_2$*,* $f(L_1) \subseteq (L_2 \cup \perp)$*;*
2. f *preserves initial state, i.e.* $f(i_1) = i_2$*;*
3. f *respects the transition relation, i.e.*

$$T_1 : s \xrightarrow{a} s' \implies T_2 : f(s) \xrightarrow{f(a)} f(s')$$

if $f(a)$ *is defined, and* $f(s) = f(s')$ *if* $f(a)$ *is undefined.*

The presence of a morphism $f : T_1 \to T_2$ indicates that T_2 can, in a way, "simulate" T_1.

It is easy to see that the identity function is a morphism of transition systems, and the composition of two morphisms is a morphism. Therefore, transition systems, together with morphisms, form a category $\mathbf{T}$. By $\mathbf{T}_L$ we denote a subcategory of $\mathbf{T}$ of transition systems with the set of labels L.

The category of the transition systems is pointed. The terminal object, denoted by *nil*, is a transition system, consisting of just one state i_{nil}, without any transitions. The terminal morphism f from a transition system T to *nil* is defined as

$$f(s) = i_{nil}$$

for all states s and is undefined on labels.

Interestingly, *nil* is also an initial object in $\mathbf{T}$. The initial morphism h from *nil* to a transition system T is defined by $h(i_{nil}) = i_T$.

2.1.1 Synchronization trees

Synchronization trees are special kinds of transition systems, that have no cycles, and consist of reachable states. Formally, a transition system $T = (S, i, L, \rightarrow)$ is a synchronization tree if

1. Every state $s \in S$ is reachable, that is, there exists a finite sequence $i \xrightarrow{v}_* s$ in T from the initial state i to s;
2. $s \xrightarrow{v}_* s \implies v = []$, i.e., if s is reachable from s via a string of labels, then v is empty;
3. If $s' \xrightarrow{a}_* s$ and $s'' \xrightarrow{b}_* s$, then $s' = s''$ and $a = b$.

By isolating synchronization trees from the wider category of transition systems, we obtain the category $\mathbf{S}$ of synchronization trees, which is a full subcategory of $\mathbf{T}$.

2.2 Event structures

Event structures are to true concurrency what synchronization trees are to interleaving concurrency. It is thus the case that we chose two main representative models of concurrency: transition systems, and event structures.

Definition 3 (Event structure)**.** *A* (labelled) event structure *is a tuple* $(E, \leq, \#, l, L)$ *consisting of*

- *a carrier E, which is the set of* events*;*
- *a* causal dependency *partial order $\leq$ on E;*
- *a* conflict *symmetric irreflexive relation $\#$ on E;*
- *a labelling function $l : E \rightarrow L$ (sometimes, when it unambiguous, we will leave out the labeling set L in the definition.*

satisfying the following laws:

- *For each $e \in E$, the set of predecessors $\{x \mid x \leq e\}$ of e is finite;*
- *For each $e, e', e'' \in E$, $e\#e' \wedge e' \leq e'' \implies e\#e''$*

For an event structure we also define a *concurrency* relation $co \subseteq E \times E$: two events are said to be concurrent if they are not causally dependant or in conflict

$$e\,co\,e' \iff \neg(e \leq e') \wedge \neg(e' \leq e) \wedge \neg(e\#e')$$

Event structures can also be defined in terms of special Petri nets, see [3] for details. The alternative definition allows for convenient graphical representation of event structures, and for definitions of unfoldings of Petri nets.

The concept of a "state" in an event structure is formulated as a *configuration.*

Definition 4 (Configuration)**.** *A configuration x of an event structure $(E, \leq, \#)$ is a subset of events $x \subseteq E$, such that*

- *x is conflict free: $\forall e, e' \in x$, $\neg(e\#e')$;*
- *x is downwards closed: $\forall e, e'$, $e \leq e' \wedge e' \in x \implies e \in x$.*

Note that because of our restriction on the sets of predecessors of the events, a configuration is necessary finite. We also introduce notation for a *local configuration of event* e:

$$\ulcorner e \urcorner = \{e' \in E \mid e' \leq e\}$$

Reader can notice that Pratt's concept of a *pomset* (partially-ordered multiset) [4] is equivalent to the concept of a configuration of a labelled event structure. In addition, both pomsets and configurations can be viewed as special event structures themselves: event structures without conflict.

Event structures form a category, in which the morphisms are defined as following:

Definition 5. *A morphism of event structures $(E_1, \leq_1, \#_1, l_1 : E_1 \to L_1)$ and $(E_2, \leq_2, \#_2, l_2 : E_2 \to L_2)$ is a partial function f such that*

- *$f(L_1) \subseteq L_2$;*
- *$f(E_1) \subseteq E_2$;*
- *f preserves labels, i.e. $l_2 \circ f = f \circ l_1$;*
- *f preserves configurations, i.e. if x is a configuration of E_1, then $f(x)$ is a configuration of E_2, and f is injective on x: if $e, e' \in x$ and $f(e) = f(e')$ are both defined, then $e = e'$.*

We denote the category of event structures by $\mathbf{E}$.

2.3 Subcategory of paths

Each category of models of concurrency considered here is equipped with a notion of a path. A (computational) path of an object X should represent a run of X. The path, therefore, has to contain the information about the "history" of the run and information about the resulting state of X. For transition systems (and synchronization trees), computational paths are usually represented by finite ordered sequences of transitions that became (or can become) active during the run. Categorically speaking, let $\mathbf{Bran}_L$ denote the (full) subcategory of $\mathbf{T}_L$ containing single-branch transition systems and morphism between them. A path of X then can be represented by a morphism $p : P \to X$, where P is an object of $\mathbf{Bran}_L$. The information that should be contained in the computational path is thus divided into two parts. The object P is responsible for size of the path and the labels of the transitions. The morphism p embeds the successive events of P into a branch of successive events of X; therefore, p is responsible for the shape of the path, as it is in X.

In case of event structures, the computational states are represented by configurations, which are equivalent to pomsets. The paths subcategory of $\mathbf{E}_L$ is thus the category $\mathbf{Pom}_L$ of pomsets over L.

In the general setting the category containing path objects will be called a *path category*, and usually we will denote it by $\mathbf{P}$. It is worth noting that in our cases (and in other examples), the inclusion $\mathbf{P} \hookrightarrow \mathbf{M}$ of a path category $\mathbf{P}$ into a category of models $\mathbf{M}$ is full. Further in the text it will be useful to assume that $\mathbf{P}$ and $\mathbf{M}$ shares the same common initial object (as it is the case in our examples); however, this is not necessary for all the theorems stated below.

2.4 Relations between categories

A familiar notion of an *unfolding* give rises to a functor U from $\mathbf{T}$ to $\mathbf{S}$. If $X = (S, s_0, L, \to_X)$ is a transition system, then $U(X)$ is a synchronization tree, which states are sequences of the form $\langle s_0, a_1, \ldots, a_n, s_n\rangle$, where $s_i \in S$ and $a_i \in L$, and $s_i \xrightarrow{a_{i+1}}_X s_{i+1}$. There is a transition labeled by a_{n+1} between $\langle s_0, a_1, \ldots, a_n, s_n\rangle$ and $\langle s_0, a_1, \ldots, a_n, s_n, a_{n+1}, s_{n+1}\rangle$ if $s_n \xrightarrow{a_{n+1}}_X s_{n+1}$. It is easy to verify that $U(X)$ is a transition system with the initial state $\langle s_0\rangle$.

If $g : T_1 \to T_2$ is a morphism of transition systems, then $U(g)$ is defined as

$$U(g)\langle s_0, a_1, \ldots, a_n, s_n\rangle = \langle g(s_0), g(a_1), \ldots, g(a_n), g(s_n)\rangle$$

We can easily verify that $U(g)$ is a morphism in $\mathbf{S}$.

Furthermore, U is a coreflector; so, $\mathbf{S}$ is a coreflective subcategory of $\mathbf{T}$.

Theorem 1. *The unfolding functor U is a right adjoint to the inclusion functor $\mathcal{I} : \mathbf{S}_L \to \mathbf{T}_L$.*

Proof. First of all, we define a unit of the adjunction, natural transformation $\eta : 1_{\mathbf{S}} \to U \circ \mathcal{I}$. For a synchronization tree X, $\eta_X(s)$ is defined to be the unique path in X from the initial state s_0 to s.

Assume then that $f : X \to U(Y)$ is a morphism in $\mathbf{S}$. There exists a morphism $g : \mathcal{I}(X) \to Y$ in $\mathbf{T}$, that makes the following diagram commute in $\mathbf{S}$:

$$\begin{tikzcd} U(\mathcal{I}(X)) \arrow[r, "U(g)"] & U(Y) \\ X \arrow[u, "\eta_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

Define $g(s)$ to be the last element of the sequence $f(s)$. We can verify that g is indeed a morphism:

1. $g(i_0^X) = \text{last}(f(i_0^X)) = \text{last}(\langle i_0^Y \rangle) = i_0^Y$, where i_0^X is the initial state of X, and i_0^Y is the initial state of Y.

2. $X : s_1 \xrightarrow{a} s_2 \implies U(Y) : f(s_1) \xrightarrow{a} f(s_2)$. By the definition of U, this implies that $\text{last}(f(s_1)) \xrightarrow{a} \text{last}(f(s_2))$, i.e. $g(s_1) \xrightarrow{a} g(s_2)$.

Now it remains to prove that $f = U(g) \circ \eta_X$. We do that by induction of $f(s)$.

- Base case: $f(s) = \langle i_0^Y \rangle$. Then $s = i_0^X$ and $U(g)(\eta_X(i_0^X)) = U(g)(\langle i_0^X \rangle) = \langle g(i_0^X) \rangle = \langle \text{last}(f(i_0^X)) \rangle = \langle i_0^Y \rangle$.

- Inductive case: $f(s) = \langle i_0^Y, \dots, s_n \rangle$. Then $U(g)(\eta_X(s)) = U(g)(\langle i_0^X, \dots, s', a, s \rangle)$, where $\langle i_0^X, \dots, s', a, s \rangle$ is the unique path from i_0^X to s. Hence, $U(g)(\langle i_0^X, \dots, s \rangle) = \langle g(i_0^X), \dots, g(s'), a, g(s) \rangle$. By induction hypothesis, this is the same as $f(s') + +\langle a, g(s) \rangle = f(s') + +\langle a, \text{last}(f(s)) \rangle = f(s') + +\langle a, s_n \rangle = f(s)$

$\square$

Apart from the relation between **S** and **T**, we can establish an interesting relation between **S** and **Bran**, it's subcategory of paths.

Theorem 2. *Both inclusions* $\mathbf{Bran}_L \hookrightarrow \mathbf{S}_L$ *and* $\mathbf{Pom}_L \hookrightarrow \mathbf{E}_L$ *are dense.*

Proof. First we show that $\mathbf{Bran}_L \hookrightarrow \mathbf{S}_L$ is dense. An object T $\mathbf{S}_L$ is the initial cocone over all the objects $[s]$, where $s \in T$ and $[s]$ is the unique path from i_T to s. To see this, consider another object T' of $\mathbf{S}_L$ that is a cocone over the same diagram.

By $p_{[s]}$ we denote the unique morphism from $[s]$ to T, and by $p'_{[s]}$ we denote the morphism from $[s]$ to T'.

$$\begin{tikzcd} {[s']} \arrow[r, "p_{[s']}"] & T & {[s']} \arrow[r, "p'_{[s']}"] & T' \\ {[s]} \arrow[u, hook] \arrow[ur, "p_{[s]}"'] & & {[s]} \arrow[u, hook] \arrow[ur, "p'_{[s]}"'] & \end{tikzcd} \tag{1}$$

Then we can define a morphism $f : T \to T'$ as

$$f(x) = p'_{[x]}(x)$$

We can verify that it is indeed the morphism of synchronization trees:

- $f(i_T) = p'_{[i_T]}(i_T) = i_{T'}$, since p' is a morphism;

- If there is a step $s \xrightarrow{a} s'$ in T, then there is an inclusion $[s] \hookrightarrow [s']$ that commutes with $p_{[s]}, p_{[s']}$ and $p'_{[s]}, p'_{[s']}$ (as in 1). So, $f(s) = p'_{[s]}(s) = p'_{[s']}(s)$. Because $p'_{[s']}$ is a morphism, we have $p'_{[s']}(s) \xrightarrow{a} p'_{[s']}(s')$ in T', that is $f(s) \xrightarrow{a} f(s')$.

Because the inclusions $p'_{[x]}$ are unique, f is unique as well.
Virtually the similar argument can be made for event structures. $\square$

3 Open maps and bisimulation

Let's return to our general setting. Let **M** be a category of models and **P** be a path subcategory of **M**. We shall restrict our attention to a special class of morphisms of **M**, that satisfy as certain path-lifting property.

Definition 6 (**P**-open morphism)**.** *A morphism $f : X \to Y$ is called* **P***-open if for all path objects $P, Q \in \mathbf{P}$ and for all morphisms p, m, q, such that the following diagram commutes*

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "m"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & Y \end{tikzcd}$$

There exist a path $p' : Q \to X$, such that both of the "triangles" in the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "m"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] \arrow[ur, "p'"] & Y \end{tikzcd}$$

i.e., $p' \circ m = p$ and $f \circ p' = q$.

In the category of transition systems, **P**-open morphisms correspond to an interesting and familiar class of morphisms:

Proposition 1. $\mathbf{Bran}_L$*-open morphism in* **T** *are exactly bounded label-preserving*[1] *morphisms (zig-zag morphisms).*

Proof. Suppose $f : X \to Y$ is a $\mathbf{Bran}_L$-open morphism in **T**. Suppose s is a reachable state of X, and $Y : f(s) \xrightarrow{a} s'$.

Since s is reachable, there is an inclusion $p : P \hookrightarrow X$ of a path object P:

$$i_X \xrightarrow{a_1} s_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} s$$

into X.

Because s' is reachable in Y, there is also an inclusion $q : Q \hookrightarrow Y$ of a path object Q:

$$f(i_X) = i_Y \xrightarrow{a_1} f(s_1) \xrightarrow{a_2} \dots \xrightarrow{a_n} f(s) \xrightarrow{a} s'$$

into Y.

Let $k : P \to Q$ be a restriction of f to P. It is easy to see that $q \circ k = f \circ p$, i.e. that the following diagram commutes.

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "k"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & Y \end{tikzcd}$$

[1] Morphisms that are identities on the set of labels.

Because f is an open morphism, there is a morphism $q' : Q \to X$, that makes the following diagram commute

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "k"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] \arrow[ur, "q'"] & Y \end{tikzcd}$$

Let z be the image of $s' \in Q$ under q'; then $f(z) = s'$. Furthermore, $q'(f(s)) = s$, because of the commutativity of the left "triangle". Since $Q : f(s) \xrightarrow{a} s'$, we get $X : q'(f(s)) = s \xrightarrow{a} q(s') = z$. Therefore, f is a bounded morphism.

Conversely, let f be a bounded morphism. Let P, Q be object of $\mathbf{Bran}_L$, s.t. the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "k"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] & Y \end{tikzcd}$$

Ultimately, Q is just a finite extension of P. By applying the condition of the bounded morphism finitely many times, we can construct a morphism $q' : Q \to Y$, s.t. $q' \circ k = p$ and $f \circ q' = q$.

□

In the case of event structures, the correspondence might not be that straightforward, but we can still characterize $\mathbf{Pom}_L$-open morphisms.

Proposition 2. *If $f : E \to E'$ is a $\mathbf{Pom}_L$-open morphism of event structures, and x is a configuration of E, then the restriction of f to x is an isomorphism of pom-sets x and fx.*

Proof. Assume $f : E \to E'$ is a $\mathbf{Pom}_L$-open morphism of event structures, and x is a configuration of E. Then, if we view x as an event structure, there is an inclusion $x \hookrightarrow E$. Similarly for fx, we obtain an inclusion $fx \hookrightarrow E'$. We obtain a commuting square:

$$\begin{tikzcd} x \arrow[r, hook] \arrow[d, "f"'] & E \arrow[d, "f"] \\ fx \arrow[r, hook] & E' \end{tikzcd}$$

By the openness of f, there is a morphism $q : fx \to E$ that makes the diagram commute

$$\begin{tikzcd} x \arrow[r, hook] \arrow[d, "f"'] & E \arrow[d, "f"] \\ fx \arrow[r, hook] \arrow[ur, "q"] & E' \end{tikzcd}$$

By definition of a morphism of event structures, $f : x \to fx$ is already injective (and surjective, by definition); the commutativity of the upper square makes sure that f is an isomorphism of x and fx.

□

We can easily check that the class of **P**-open morphisms includes all isomorphisms, and is closed under composition.

Let f be an isomorphism, and let p, q, m be morphisms such that $f \circ p = q \circ m$

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "m"'] & X \arrow[d, "f"] \\ Q \arrow[r, "q"'] \arrow[ur, "f^{-1} \circ q"] & Y \end{tikzcd}$$

Then the morphism $f^{-1} \circ q$ clearly satisfies the conditions of definition 6.

Let f and g be **P**-open. Then $g \circ f$ is **P**-open as well.

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[dd, "m"'] & X \arrow[d, "f"] \\ & Y \arrow[d, "g"] \\ Q \arrow[r, "q"'] \arrow[uur, dotted, "p''"] \arrow[ur, dotted, "p'"] & Z \end{tikzcd}$$

Let $p : P \to X, q : Q \to Z$, and $m : P \to Q$ be morphisms such that $q \circ m = (g \circ f) \circ p$. Then, by openness of g there is a morphism $p' : Q \to Y$, such that $p' \circ m = f \circ p$ and $g \circ p' = q$. Then, by openness of f there is another morphism $p'' : Q \to X$, such that the both required "triangles" commute.

The notion of **P**-openness can also be used to define a generalized notion of bisimulation. We say that two objects $X_1, X_2 \in \mathbf{M}$ are **P**-bisimilar if they are connected by a span of **P**-open maps.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_1 & & X_2 \end{tikzcd}$$

In the specific cases of transition systems and event structures we can obtain quite satisfactory characterizations for the notion of **P**-bisimilarity.

Theorem 3. *Two transition systems are* $\mathbf{Bran}_L$*-bisimilar iff they are (strongly) bisimilar in the sense of Milner.*

Proof. $\Rightarrow$: if two objects X_1, X_2 are connected to another object Y by $\mathbf{Bran}_L$-open morphisms, then they are connected to Y via bounded morphisms. In that case, the graphs of bounded morphisms are bisimulations; by the symmetry and transitivity of bisimulation, X_1 and X_2 are bisimilar.

$\Leftarrow$: Let R be a bisimulation relation between X_1 and X_2. We construct a transition system Y as follows:

- The collection of states of Y is R itself (viewed as a set).
- There is a transition $(s_1, s_2) \xrightarrow{a} (s_1', s_2')$ in Y if $s_1 \xrightarrow{a} s_1'$ in X_1 and $s_2 \xrightarrow{a} s_2'$ in X_2.

Clearly, the projections $\pi_1 : Y \to X_1$ and $\pi_2 : Y \to X_2$ are morphisms; it is also easy to see that they are bounded morphisms. Because of proposition 1, they are also $\mathbf{Bran}_L$-open. Thus, X_1 and X_2 are $\mathbf{Bran}_L$-bisimilar. $\square$

In order to characterize $\mathbf{Pom}_L$-bisimilarity between event structures, let's recall the definition of a *history-preserving bisimulation*:

Definition 7. *A* history-preserving bisimulation *between two event structures E_1 and E_2 is a set H of triples (x_1, f, x_2), where x_1 and x_2 are configurations of E_1 and E_2 respectively, and f is an isomorphism between them; furthermore, H has to satisfy the following properties: $(\emptyset, \emptyset, \emptyset) \in H$, and whenever $(x_1, f, x_2) \in H$, then*

1. *If $E_1 : x_1 \xrightarrow{a} x_1'$, then $E_2 : x_2 \xrightarrow{a} x_2'$ for some $(x_1', f', x_2') \in H$ with $f \subseteq f'$.*
2. *If $E_2 : x_2 \xrightarrow{a} x_2'$, then $E_1 : x_1 \xrightarrow{a} x_1'$ for some $(x_1', f', x_2') \in H$ with $f \subseteq f'$.*

The bisimulation is called strong *if it further satisfies*

1. *$(x_1, f, x_2) \in H$ and $x_1' \subseteq x_1$ for a configuration x_1' of E_1 implies $(x_1', f', x_2') \in H$ for some $f' \subseteq f$ and $x_2' \subseteq x_2$;*
2. *$(x_1, f, x_2) \in H$ and $x_2' \subseteq x_2$ for a configuration x_2' of E_2 implies $(x_1', f', x_2') \in H$ for some $f' \subseteq f$ and $x_1' \subseteq x_1$.*

Theorem 4. *If two event structures are $\mathbf{Pom}_L$-bisimilar, then they are strongly history-preserving bisimilar.*

Proof. Assume that X_1 and X_2 are $\mathbf{Pom}_L$-bisimilar, that is, they are connected via a span of open morphism from some object Y. It is suffices to show that X_1, Y and X_2, Y are pairwise strong history-preserving bisimilar.

Let $f : Y \to E_1$ be a $\mathbf{Pom}_L$-open morphism. Then we take H to be a set of all triples (x, f', fx), where x is a configuration of Y, and f' is the restriction of f on x; from proposition 2 we get that f' is an isomorphism, thus H is well-defined.

It is clear that $(\emptyset, \emptyset, \emptyset) \in H$. The first condition of history-preserving bisimulation is fulfilled automatically, in the virtue of f being a morphism. We can show that the second condition of history-preserving bisimulation holds as well. Let $(x, f', fx) \in H$, and let $E_1 : fx \xrightarrow{a} y$ for some configuration y. The inclusion $fx \subseteq y$ can be viewed as a morphism of pomsets $fx \hookrightarrow y$. By m let us denoted the composition of morphisms $x \xrightarrow{f'} fx \hookrightarrow y$. We obtain the following commutative diagram:

$$\begin{tikzcd} x \arrow[r, hook] \arrow[d, "m"'] & E \arrow[d, "f"] \\ y \arrow[r, hook] & E' \end{tikzcd}$$

Since f is $\mathbf{Pom}_L$-open, there is a morphism $q : y \to E$, that makes the diagram commute

$$\begin{tikzcd} x \arrow[r, hook] \arrow[d, "m"'] & E \arrow[d, "f"] \\ y \arrow[r, hook] \arrow[ur, "q"] & E' \end{tikzcd}$$

Then, we take x' to be $q(m(x))$. By the commutativity of the lower triangle, $f(x') = y$. Therefore, the restriction f'' of f on x' is an isomorphism between

x' and y. Hence, $(x', f'', y) \in H$ and the second condition of history-preserving bisimulation is verified.

It remains to show that H is a *strong* history-preserving bisimulation; however, the *strongness* conditions are automatically satisfied due to our choice of isomorphisms in H.

□

However, for the relation of **P**-bisimilarity to be suitable for modeling wide range of bisimulation-like relations, clearly, it has to be reflexive, symmetric, and transitive. The symmetry arises from the nature of the spans, and reflexivity holds because the identity morphisms are **P**-open. However, it is also transitive if the category **M** has pullbacks, due to the following fact.

Theorem 5. *Pullbacks of* $\mathbf{P}$*-open morphisms are* $\mathbf{P}$*-open.*

Proof. Assume we have the following pullback square

$$\begin{tikzcd} & C \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ A \arrow[dr, "f_1"'] & & B \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

where f_1,f_2 are **P**-open. It's our aim to show that g_1 and g_2 are open as well. Assume, then, that we have another commuting square with g_1, where P, Q are from **P**

$$\begin{tikzcd} & P \arrow[r, "p"] \arrow[dl, "m"'] & C \arrow[dl, "g_1"] \arrow[dr, "g_2"] & \\ Q \arrow[r, "q"] & A \arrow[dr, "f_1"'] & & B \arrow[dl, "f_2"] \\ & & X & \end{tikzcd}$$

Because both squares commute, the big square commutes as well, i.e. $f_2 \circ g_2 \circ p = f_1 \circ q \circ m$. From the **P**-openness of f_2, we obtain a morphism $k : Q \to B$, such that

$$f_2 \circ k = f_1 \circ q$$

and

$$k \circ m = g_2 \circ p$$

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "m"'] & C \arrow[r, "g_2"] \arrow[d, "k"'] & B \arrow[d, "f_2"] \\ Q \arrow[r, "q"'] \arrow[urr, dashed] & A \arrow[r, "f_1"'] & X \end{tikzcd}$$

Because C is the pullback of f_1, f_2, there exists a unique morphism $h : Q \to C$, such that $k = g_2 \circ h$, and $q = g_1 \circ h$.

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "m"'] & C \arrow[r, "g_2"] \arrow[d, "g_1"] & B \arrow[d, "f_2"] \\
Q \arrow[r, "q"'] \arrow[ur, dashed, "h"] \arrow[urr, dashed, "k"] & A \arrow[r, "f_1"'] & X
\end{tikzcd} \tag{2}$$

The morphism h is a good candidate for a morphism $Q \to C$ that we have to find. We already have the commutativity of the lower square from the pullback laws, it only remains to show that $h \circ m = p$. To prove this, consider the commuting square

$$\begin{tikzcd}
P \arrow[r, "g_2 \circ p"] \arrow[d, "q \circ m"'] & B \arrow[d, "f_2"] \\
A \arrow[r, "f_1"] & X
\end{tikzcd}$$

There exists an obvious morphism satisfying the universal property of C: the morphism p. If we manage to show that $h \circ m$ also makes the diagram below commute, then we are done, because such morphism is unique.

$$\begin{tikzcd}
P \arrow[dr, "h \circ m"] \arrow[drr, bend left, "g_2 \circ p"] \arrow[ddr, bend right, "q \circ m"'] & & \\
& C \arrow[r, "g_2"] \arrow[d, "g_1"'] & B \arrow[d, "f_2"] \\
& A \arrow[r, "f_1"] & X
\end{tikzcd}$$

Firstly, $g_1 \circ (h \circ m) = q \circ m$ because of the commutativity of the lower triangle in (2). Secondly, $g_2 \circ p = k \circ m$ (because of the commutativity of one o the triangles in (2)), and $k \circ m = g_2 \circ h \circ m$.

Therefore, $h \circ m = p$.

The **P**-openness of g_2 can be proven the same way.

$\square$

Finally, we can see why the statement of theorem 5 implies transitivity of **P**-openness. Consider three objects A, B, and C, A and B being related by a span of **P**-open morphisms via Y_1, and B and C being related by a span of **P**-open morphisms via Y_2, as in the picture

$$\begin{tikzcd}
Y_1 \arrow[d, "f_1"'] \arrow[dr, "f_2"] & & Y_2 \arrow[dl, "g_1"'] \arrow[d, "g_2"] \\
A & B & C
\end{tikzcd}$$

We can take a pullback of f_2 and g_1:

$$\begin{tikzcd} & Y_1 \times_B Y_2 \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ Y_1 \arrow[d, "f_1"'] \arrow[dr, "f_2"] & & Y_2 \arrow[d, "g_2"] \arrow[dl, "g_1"'] \\ A & B & C \end{tikzcd}$$

Then the morphism h_1 and h_2 are **P**-open; hence, $f_1 \circ h_1$ and $g_2 \circ h_2$ are **P**-open. As a result, A and B are connected by a span of open morphisms and are **P**-bisimilar.

Additionally, we will need the following theorem, which is a simple consequence of the definitions.

Theorem 6. *Let* $\mathbf{P}$ *be a subcategory of* $\mathbf{M}$*, and* $\mathbf{M}$ *be a full subcategory of* $\mathbf{N}$*. A morphism is* $\mathbf{P}$*-open in* $\mathbf{M}$ *iff it is* $\mathbf{P}$*-open in* $\mathbf{N}$*.*

4 Presheaves for concurrency

A presheaf over $\mathbf{P}$ is a functor $F : \mathbf{P}^{op} \to \mathbf{Set}$. The category of presheaves over $\mathbf{P}$ and natural transformations, denoted as $[\mathbf{P}^{op}, \mathbf{Set}]$, is a topos. The notion of an open map in a topos ([1]) corresponds to the notion of **P**-open morphism. In this section we will study this, and other connections between presheaves and models of concurrency.

4.1 Canonical inclusion

Returning to our general setting, with a category of models $\mathbf{M}$, and a subcategory of path objects $\mathbf{P} \hookrightarrow \mathbf{M}$, we define a *canonical functor*, sending objects from $\mathbf{M}$ to presheaves over $\mathbf{P}$:

$$C(X) = Hom_M(-, X)$$

$C(X)$ sends a path object P to a set $Hom_M(P, X)$ and a morphism $m : P \to Q$ to a function $- \circ m : Hom_M(Q, X) \to Hom_M(P, X)$. C sends morphism $A \to B$ to natural transformations $C(A) \Rightarrow C(B)$, defined as

$$C(f : A \to B)_P = f \circ -$$

So, $C(f)_P(g : P \to A) = f \circ g : P \to B$. The reader may notice a strong similarity between the canonical functor C and the Yoneda functor $\mathcal{Y}$. In fact, two concept coincide on $\mathbf{P}$ if $\mathbf{P}$ is a full subcategory of $\mathbf{M}$. Yoneda functor is full and faithful, but we cannot say the same about C, in general. However, C is be full and faithful if the inclusion $\mathbf{P} \hookrightarrow \mathbf{M}$ is dense.

Theorem 7. *Let the inclusion functor* $\mathbf{P} \hookrightarrow \mathbf{M}$ *be dense. Then* $C(X) = Hom_M(-, X) : \mathbf{P}^{op} \to \mathbf{Set}$ *is full and faithful.*

Proof. First, we show that C is full. Let f be a morphism between $C(A)$ and $C(B)$. That is, $f_P : Hom_M(P, A) \to Hom_M(P, B)$ sends an arrow $P \xrightarrow{g} A$ to an arrow $P \xrightarrow{f_P(g)} B$.

Because of the denseness, A is a limiting cocone of a diagram $F : D_A \to \mathbf{M}$, where D_A is a subcategory of $\mathbf{P}$.

$$\begin{tikzcd}
& A & \\
P_1 \arrow[ur, "\psi_1"] \arrow[r] & \cdots \arrow[r] & P_j \arrow[ul, "\psi_j"']
\end{tikzcd}$$

Using f_P on the components of the cocone with the vertex A we can define a cocone with the vertex B.

$$\begin{tikzcd}
& A & & B \\
P_1 \arrow[ur, "\psi_1"] \arrow[urrr, "f_{P_1}(\psi_1)"] \arrow[r] & \cdots \arrow[r] & P_j \arrow[ul, "\psi_j"'] \arrow[ur, "f_{P_j}(\psi_j)"']
\end{tikzcd}$$

Indeed, this produces a cocone, because of the naturality of f. To see this, consider the following part of the cocone A:

$$\begin{tikzcd}
A & & B \\
P_1 \arrow[u, "l"] \arrow[urr, "f_{P_1}(l)"] \arrow[r, "k"] & P_2 \arrow[ul, "m"'] \arrow[ur, "f_{P_2}(m)"']
\end{tikzcd}$$

We want to show that $f_{P_2}(m) \circ k = f_{P_1}(l)$. Then, from naturality of f, we get the following commutative diagram:

$$\begin{tikzcd}
Hom(P_1, A) \arrow[r, "f_{P_1}"] & Hom(P_1, B) \\
Hom(P_2, A) \arrow[u, "- \circ k"] \arrow[r, "f_{P_2}"] & Hom(P_2, B) \arrow[u, "- \circ k"']
\end{tikzcd}$$

By chasing $m \in Hom(P_2, A)$ around the diagram, we get $f_{P_1}(l) = f_{P_1}(m \circ k) = f_{P_2}(m) \circ k$.

Because A is the initial cocone, there exists a unique morphism $g : A \to B$, such that for all objects $Q \in \mathbf{P}$ and morphisms $q : Q \to A$, the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "g"] & B \\
Q \arrow[u, "q"] \arrow[ur, "f_Q(q)"']
\end{tikzcd}$$

i.e., $C(g)_Q(q) = f_Q(q)$ for all Q, q. So, C is full.

In order to see that C is faithful, consider two morphism f, g, $f \neq g$. Then $C(f)(Id) = f \circ Id = f \neq g = g \circ Id = C(g)(Id)$.

□

Because we know that both inclusions $\mathbf{Bran} \hookrightarrow \mathbf{S}$ and $\mathbf{Pom} \hookrightarrow \mathbf{E}$ are dense (c.f. theorem 2), we immediately obtain the following:

Proposition 3. *1. The canonical functor from* $\mathbf{S}$ *to* $[\mathbf{Bran}^{op}, \mathbf{Set}]$ *is a full embedding.*

2. The canonical functor from $\mathbf{E}$ *to* $[\mathbf{Pom}^{op}, \mathbf{Set}]$ *is a full embedding.*

Transition systems $\mathbf{T}$ are not embedded in $[\mathbf{Bran}^{op}, \mathbf{Set}]$. For example, consider two transition systems T_1 (fig. 1) and T_2 (fig. 2). Presheaves $C(T_1)$ and $C(T_2)$ are isomorphic, while T_1 and T_2 clearly are not. Presheaves, as ways of modeling concurrency, simply do not posses enough information to distinguish between individual states, as it is the case between T_1 and T_2.

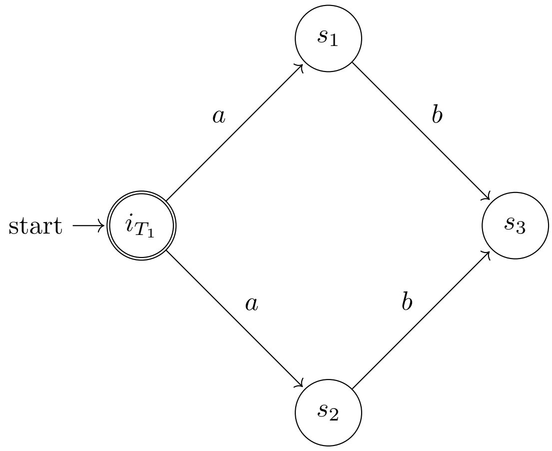

Fig. 1: Transition system T_1

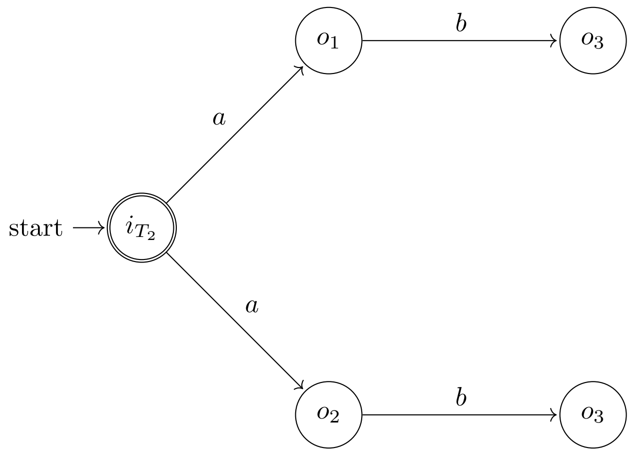

Fig. 2: Transition system T_2

However, in some sense, presheaves are richer than major models of concurrency, like transition systems, synchronization trees, or event strucutres. For example, consider the category of presheaves $[\mathbf{Bran}^{op}, \mathbf{Set}]$. It contains a full copy of $\mathbf{S}$, but there are many more presheaves than synchronization trees. A presheaf X contains the information about possible paths (values of X at objects from $\mathbf{P}$), and information on how to glue those paths together. For example, if X is a presheaf over **Bran**, and $X(P) = \{a, b\}$, $X(Q) = \{c\}$, then we know that there are two ways of computing path P in X, and only one way of computing path Q. Furthermore, if $P \xrightarrow{m} Q$ (i.e., Q is an extension of P), then $X(m)(c)$ tells us exactly how Q extends P *in* X.

Intuitively, we can view additional objects in $[\mathbf{Bran}^{op}, \mathbf{Set}]$ as "synchronization forests" – collections of synchronization trees. If X is an arbitrary presheaf over **Bran**, then $X(nil)$ need not be a singleton set, containing unique morphism from *nil*. This allows X to contain multiple initial states.

Some of the additional objects in $[\mathbf{Pom}^{op}, \mathbf{Set}]$ can be accounted for with Petri nets. Consider, for example, a presheaf X, such that $X(s) = \{a\}$ if s is a pomset containing only events labeled with a., and $X(s) = \emptyset$ otherwise. This presheaf corresponds to a Petri net consisting just of one transition a.

4.2 Presheaves and open maps

The definition of an open map in a topos applies to the category of presheaves. Viewing $\mathbf{P}$ as a subcategory of $[\mathbf{P}^{op}, \mathbf{Set}]$ (via the Yoneda embedding), it is possible to check that the definition of an open map in $[\mathbf{P}^{op}, \mathbf{Set}]$ coincides with the definition of a $\mathcal{Y}\mathbf{P}$-open morphism.

In fact, this point of view provides us with an instance of a general setup: the category of presheaves is the category of models, and $\mathcal{Y}\mathbf{P}$ is the subcategory of path objects.

We can establish a nice relation between open morphism in $\mathbf{M}$, and open maps in $[\mathbf{P}^{op}, \mathbf{Set}]$.

Theorem 8. *Let $\mathbf{P}$ be a full, dense subcategory of $\mathbf{M}$, and $f : X \to Y$ be a morphism in $\mathbf{M}$. Then, f is $\mathbf{P}$-open in $\mathbf{M}$ iff $C(f)$ is $\mathcal{Y}\mathbf{P}$-open in $[\mathbf{P}^{op}, \mathbf{Set}]$.*

Proof. Because $\mathbf{P}$ is a full subcategory of $\mathbf{M}$,

$$\mathcal{Y}P = Hom_{\mathbf{P}}(-, P) = Hom_{\mathbf{M}}(-, P) = CP$$

for all path objects P in $\mathbf{P}$. That means that the following diagram (in **Cat**) commutes:

$$\begin{tikzcd} \mathbf{P} \arrow[d, hook] \arrow[dr, hook, "\mathcal{Y}"] & \\ \mathbf{M} \arrow[r, hook, "C"] & {[\mathbf{P}^{op}, \mathbf{Set}]} \end{tikzcd}$$

By the theorems 6 and 7 we obtain that a morphism is $\mathbf{P}$-open in $\mathbf{M}$ iff it is $C\mathbf{P}$-open in $[\mathbf{P}^{op}, \mathbf{Set}]$. Finally, as we have seen, $C\mathbf{P} = \mathcal{Y}\mathbf{P}$. $\square$

Unfortunately, a similar relation cannot be established between $\mathbf{P}$-bisimulation in $\mathbf{M}$, and $\mathcal{Y}\mathbf{P}$-bisimulation in $[\mathbf{P}^{op}, \mathbf{Set}]$. This is due to the fact that the presheaf category contains much more objects than $\mathbf{M}$. Particularly, consider the initial presheaf $\emptyset(-) = \emptyset$. Any arrow *from* $\emptyset(-)$ is trivially $\mathcal{Y}\mathbf{P}$-open, because there are no possible arrows *to* $\emptyset(-)$ (apart the identity arrow)! This suggests that in $[\mathbf{P}^{op}, \mathbf{Set}]$ all objects are $\mathcal{Y}\mathbf{P}$-bisimilar. In order to combat this problem, we introduce a notion of a *rooted presheaf*.

4.3 Rooted presheaves

In this section we presume that the category $\mathbf{P}$ contains the initial object I, which is also the initial object in $\mathbf{M}$.

Definition 8. *A presheaf $X : \mathbf{P}^{op} \to \mathbf{Set}$ is said to be* rooted *if $X(I)$ is a singleton set, where I is the initial object of $\mathbf{P}$.*

Clearly, rooted presheaves form a subcategory of $[\mathbf{P}^{op}, \mathbf{Set}]$. We shall denoted that subcategory as $[\mathbf{P}^{op}, \mathbf{Set}]_r$. It is also clear that the image of $\mathbf{M}$ under C lies in $[\mathbf{P}^{op}, \mathbf{Set}]_r$, together with the image of $\mathbf{P}$ under $\mathcal{Y}$. Therefore, we can easily restate theorem 8 in terms of rooted presheaves.

Theorem 9. *Let $\mathbf{P}$ be a full, dense subcategory of $\mathbf{M}$, and $f : X \to Y$ be a morphism in $\mathbf{M}$. Then, f is $\mathbf{P}$-open in $\mathbf{M}$ iff $C(f)$ is $\mathcal{Y}\mathbf{P}$-open in $[\mathbf{P}^{op}, \mathbf{Set}]_r$.*

We can also verify, that the initial object is preserved under the Yoneda embedding, restricted to the category of rooted presheaves.

Proposition 4. *If I is the initial object of $\mathbf{P}$, then $\mathcal{Y}I = Hom(-, I)$ is the initial object of $[\mathbf{P}^{op}, \mathbf{Set}]_r$.*

Proof. To show that $\mathcal{Y}I$ is initial in $[\mathbf{P}^{op}, \mathbf{Set}]_r$, consider a rooted presheaf X. By the Yoneda lemma, the set of natural transformations between $\mathcal{Y}I$ and X is isomorphic to $X(I)$. However, because X is a *rooted* presheaf, $X(I)$ is a singleton. Therefore, for rooted presheaf X there is exactly one map from $\mathcal{Y}I$ to X. □

In addition, the following theorem holds for the category of rooted presheaves.

Theorem 10.
1. *Two synchronization trees over a common labelling set are $\mathbf{Bran}_L$-bisimilar iff their images under C are $\mathcal{Y}(\mathbf{Bran}_L)$-bisimilar in the category of rooted presheaves over $\mathbf{Bran}_L$.*
2. *Two event structures over a common labelling set are $\mathbf{Pom}_L$-bisimilar iff their images under C are $\mathcal{Y}(\mathbf{Pom}_L)$-bisimilar in the category of rooted presheaves over $\mathbf{Pom}_L$.*

Currently, we lack the machinery to prove this theorem. We will prove it using a roundabout: we will define a modal logic, which characterizes our generalized notion of bisimulation for presheaves, synchronization trees, and event structures.

Proposition 5. *The category $[\mathbf{Bran}^{op}, \mathbf{Set}]_r$ is equivalent to $\mathbf{S}$.*

4.3.1 Path logic

Path formulae are defined as

$$\phi := \overline{\langle m \rangle}\phi \mid \langle m \rangle \phi \mid \neg\phi \mid \bigwedge_{i \in J} \phi_i$$

where m is a morphism from $\mathbf{P}$, and J is a (possibly infinite) indexing set. We define $\top$ as an empty conjunction, and $\bot$ as $\neg\top$.

The $\langle m \rangle$ modality is called a "forward" modality, and $\overline{\langle m \rangle}$ is a "backwards" modality.

The path formulae are interpreted over paths $p : P \to X$, where P is an path object from $\mathbf{P}$ and X is an object from $\mathbf{M}$. Specifically,

- $p \models \langle m \rangle A$, for $m : P \to P'$, if there exists a $p' : P' \to X$, such that $p' \circ m = p$, and $p' \models A$; i.e., if we can extend path p to path p' via m, such that p' satisfies A.
- $p \models \overline{\langle m \rangle} A$, for $m : P' \to P$, if $(p \circ m) \models A$.
- $p \models \bigwedge_{i \in J} A_i$ if $p \models A_i$ for all $i \in J$.
- $p \models \neg A$ if $p \not\models A$.

We say that a proposition A holds for an object X of $\mathbf{M}$ (denoted as $X \models A$), if A holds for the initial path into X: $(i : I \to X) \models A$.

We can define a notion of bisimulation for this logic, and prove a version of Hennessy-Milner theorem for it.

Definition 9 (Path bisimulation)**.** *A* path bisimulation *between two objects $X_1, X_2 \in \mathbf{M}$ is a binary relation R between paths going to X_1 and X_2 from the same domain. That is, if $(p_1, p_2) \in R$, then $p_1 : P \to X_1$ and $p_2 : P \to X_2$, for some P. Furthermore, this relation has to satisfy the following:*

1. *Initial paths are related. If i_1 is a unique path $I \to X_1$ and i_2 is a unique path $I \to X_2$, then $(i_1, i_2) \in R$.*
2. *If $(p_1, p_2) \in R$, and $p_1 = p_1' \circ m$ with m being a morphism from* $\mathbf{P}$ *(i.e., p_1 can be extended to p_1' via m),*

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[d, "m"] \arrow[dr, "p_2"] & \\ X_1 & P' \arrow[l, "p_1'"] & X_2 \end{tikzcd}$$

then there is a path p_2', such that $p_2 = p_2' \circ m$, and $(p_1', p_2') \in R$.

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[d, "m"] \arrow[dr, "p_2"] & \\ X_1 & P' \arrow[l, "p_1'"] \arrow[r, "p_2'"'] & X_2 \end{tikzcd}$$

3. *(Condition symmetric to the previous one)*
4. Condition for path bisimulation being <u>strong</u>:
   If $(p_1 : P \to X_1, p_2 : P \to X_2) \in R$ and $m : P' \to P$ is a morphism in $\mathbf{P}$*,*

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & P' \arrow[u, "m"] & X_2 \end{tikzcd}$$

then $(p_1 \circ m, p_2 \circ m) \in R$.

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & P' \arrow[u, "m"] \arrow[l, "p_1 \circ m"] \arrow[r, "p_2 \circ m"'] & X_2 \end{tikzcd}$$

We say that two objects are (strongly) path-bisimilar if there exists a (strong) path bisimulation between them.

Theorem 11. *Two objects are strongly path-bisimilar iff they satisfy the same set of path formulae.*

Proof. $\Rightarrow$ Let R be a strong path bisimulation between X_1 and X_2. We prove that X_1 and X_2 satisfy the same path formulae (denoted as $X_1 \sim X_2$) by structural induction on the formula. But first, we strengthen the induction hypothesis: for all $p_1 : P \to X$ and $p_2 : P \to X$, such that $(p_1, p_2) \in R$, for all ϕ, $p_1 \models \phi \iff p_2 \models \phi$.

1. Case $\phi = \bigwedge_{j \in J} A_j$. Trivial, by IH.
2. Case $\phi = \neg A$. Trivial, by IH.
3. Case $\phi = \langle m \rangle A$ for $m : P \to P'$. Then,

$$p_1 \models \langle m \rangle A \iff \exists g_1 : P' \to X_1, g_1 \circ m = p_1$$

   by the condition of path bisimilarity, there is a $g_2 : P' \to X_2$, such that $g_2 \circ m = p_2$ and $(g_1, g_2) \in R$. By IH, $g_2 \models A$, which implies that $p_2 \models \langle m \rangle A$. The other direction is proved similarly.
4. Case $\phi = \overline{\langle m \rangle} A$ for $m : P' \to P$. Then, $p_1 \models \phi \iff (p_1 \circ m) \models A$. By the strongness condition of path bisimulation, $(p_1 \circ m, p_2 \circ m) \in R$. By IH, $(p_2 \circ m) \models A$, so $p_2 \models A$. The other direction is proved similarly.

$\Leftarrow$ We define the relation $R = \{(p_1, p_2) \mid \forall A.(p_1 \models A \iff p_2 \models A)\}$, and show it to be a strong path bisimulation.

1. Clearly, $(i_1, i_2) \in R$, where i_1 and i_2 are unique arrows from the initial object.
2. Assume $(p_1, p_2) \in R$, and the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & P' \arrow[u, "m"] \arrow[l, "p_1'"] & X_2 \end{tikzcd}$$

   Assume, for a contradiction, that there is no such p_2', s.t. $p_2' \circ m = p_2$ and $(p_1', p_2') \in R$. If there is no morphism $q : P' \to X_2$ that makes the diagram commute, then $p_1 \models \langle m \rangle \top$ and $p_2 \not\models \langle m \rangle \top$. This contradicts the assumption that $(p_1, p_2) \in R$. Therefore, we can assume that there is a non-empty collection of morphism $\{q_j \mid j \in J\}$, that make the diagram commute. None of them are in relation to p_1', so, by the definition of R, there is a formula Q_j for each q_j in that collection, such that $p_1' \models Q_j$ and $q_j \not\models Q_J$. Then, $p_1 \models \langle m \rangle \bigwedge_{j \in J} Q_j$, but $p_2 \not\models \langle m \rangle \bigwedge_{j \in J} Q_j$, which is a contradiction

3. The third condition is proven similarly.

4. Assume that $(p_1, p_2) \in R$ and $m : P' \to P$. Furthermore, assume for a contradiction, that $(p_1 \circ m, p_2 \circ m) \notin R$. Then there is a formula A, such that $p_1 \circ m \models A$ and $p_2 \circ m \not\models A$. Then, clearly, $p_1 \models \overline{\langle m \rangle} A$ and $p_2 \not\models \overline{\langle m \rangle} A$.

$\square$

There is an intricate relation between **P**-bisimulation and strong path bisimulation.

Theorem 12. *Let* **P** *be a subcategory of* **M**. *If objects* X_1, X_2 *of* **M** *are* **P***-bisimilar, then they are strong path bisimilar.*

Proof. Since X_1, X_2 are **P**-bisimilar, they are connected by a span of **P**-open morphisms from a common object X.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_1 & & X_2 \end{tikzcd}$$

Then let R be a set of paths that factors through X and f and g.

$$\begin{tikzcd} & P \arrow[d, "p"] & \\ & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_1 & & X_2 \end{tikzcd}$$

Formally, $(p_1 : P \to X_1, p_2 : P \to X_2) \in R$ iff $p_1 = f \circ p$ and $p_2 = g \circ p$ for some path $p : P \to X$.

We can verify that R is a strong path bisimulation.

1. Clearly, $(i_1, i_2) \in R$, where i_1, i_2 are unique morphisms from the initial object.

2. Let $(f \circ p, g \circ p) \in R$, and $q : Q \to X_1$ is a path such that $f \circ p = q \circ m$.

$$\begin{tikzcd} & P \arrow[dl, "m"'] \arrow[d, "p"] & \\ Q \arrow[d, "q"'] & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_1 & & X_2 \end{tikzcd}$$

Then, because f is **P**-open, there exists a morphism $q' : Q \to X$, such that $f \circ q' = q$. So, $q \circ m = f \circ q' \circ m$.

$$\begin{tikzcd}
& P \arrow[dl, "m"'] \arrow[d, "p"] & \\
Q \arrow[r, dotted, "q'"] \arrow[d, "q"'] & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
X_1 & & X_2
\end{tikzcd}$$

By definition of R, we get that $(q, g \circ q') = (f \circ q', g \circ q') \in R$.

3. The other case is proved symmetrically.
4. If $p : P \to X$ is a path, $(f \circ p, g \circ p) \in R$, and $m : P' \to P$ is a morphism in $\mathbf{P}$, then $(f \circ p \circ m, g \circ p \circ m)$ is in R by construction.

$\square$

An important question we may ask next, if the existence of strong path bisimulation between objects imply **P**-bisimilarity. As it turns out, it does for all the concrete models we've considered, and for rooted presheaves models.

Theorem 13. *Let X_1, X_2 be two rooted presheaves from $[\mathbf{P}^{op}, \mathbf{Set}]_r$. There is a strong path bisimulation between X_1 and X_2 iff X_1 and X_2 are $\mathbf{P}$-bisimilar.*

Proof. $\Leftarrow$ by theorem 12.

$\Rightarrow$ Let R_0 be a strong path bisimulation relation between X_1 and X_2. We can construct a presheaf $R \hookrightarrow X_1 \times X_2$, by defining

$$R(P) = \{(\tilde{p_1}, \tilde{p_2}) \mid p_1 : \mathcal{Y}P \to X_1, p_2 : \mathcal{Y}P \to X_2, (p_1, p_2) \in R_0\}$$

where $\tilde{}$ is the Yoneda isomorphism between $Hom_{[\mathbf{P}^{op}, \mathbf{Set}]}(\mathcal{Y}P, X)$ and $X(P)$. R is defined on morphisms as

$$R(m)(\tilde{p_1}, \tilde{p_2}) = (p_1 \tilde{\circ} \mathcal{Y}m, p_2 \tilde{\circ} \mathcal{Y}m)$$

There are natural transformations $\pi_1 : R \to X_1$ and $\pi_2 : R \to X_2$, which correspond to component-wise projections. We claim that they are **YP**-open.

Assume the following diagram commutes

$$\begin{tikzcd}
\mathcal{Y}P \arrow[r, "p"] \arrow[d, "m"'] & R \arrow[d, "\pi_1"] \\
\mathcal{Y}Q \arrow[r, "q"'] & X_1
\end{tikzcd}$$

i.e., $\pi_1 \circ p = q \circ m$. Via the Yoneda lemma, this is the same as saying $\tilde{p} = (\tilde{p_1}, \tilde{p_2})$, and $\tilde{p_1} = \widetilde{q \circ m}$. Because $\tilde{}$ is an isomorphism, it is the case that $p_1 = q \circ m$.

$$\begin{tikzcd}
& \mathcal{Y}P \arrow[dl, "p_1"'] \arrow[d, "m"] \arrow[dr, "p_2"] & \\
X_1 & \mathcal{Y}Q \arrow[l, "q"] & X_2
\end{tikzcd}$$

By strong path bisimulation, there is a morphism $q_2 : \mathcal{Y}Q \to X_2$, such that $q_2 \circ m = p_2$, and $(q, q_2) \in R_0$.

$$\begin{tikzcd}
 & \mathcal{Y}P \arrow[dl, "p_1"'] \arrow[d, "m"] \arrow[dr, "p_2"] & \\
X_1 & \mathcal{Y}Q \arrow[l, "q"] \arrow[r, "q_2"'] & X_2
\end{tikzcd}$$

By definition of R, it is the case that $(\tilde{q}, \tilde{q_2}) \in R(Q)$. By Yoneda lemma, there is a morphism $k = \widetilde{(\tilde{q}, \tilde{q_2})} : \mathcal{Y}Q \to R$.

$$\begin{tikzcd}
\mathcal{Y}P \arrow[r, "p"] \arrow[d, "m"'] & R \arrow[d, "\pi_1"] \\
\mathcal{Y}Q \arrow[r, "q"'] \arrow[ur, "k"] & X_1
\end{tikzcd}$$

To say that $q = \pi_1 \circ \widetilde{(\tilde{q}, \tilde{q_2})}$ is to say that $\tilde{q} = (\pi_1)_Q(\widetilde{\widetilde{(\tilde{q}, \tilde{q_2})}})$, which is true by the definition of π_1.

Because $\mathcal{Y}$ is a full embedding, $m = \mathcal{Y}s$ for some $s : P \to Q$. To show that $k \circ m = p$, consider the following naturality diagram:

$$\begin{tikzcd}
Hom(\mathcal{Y}Q, R) \arrow[r, leftrightarrow, "\tilde{\cdot}"] \arrow[d, "- \circ m = - \circ \mathcal{Y}s"'] & R(Q) \arrow[d, "R(s)"] \\
Hom(\mathcal{Y}P, R) \arrow[r, leftrightarrow, "\tilde{\cdot}"] & R(P)
\end{tikzcd}$$

Since $k \in Hom(\mathcal{Y}Q, R)$, we get that $\widetilde{k \circ m} = R(s)(\tilde{k})$. But

$$R(s)(\tilde{k}) = R(s)(\widetilde{\widetilde{(\tilde{q}, \tilde{q_2})}}) = R(s)(\tilde{q}, \tilde{q_2}) = (\widetilde{q \circ \mathcal{Y}s}, \widetilde{q_2 \circ \mathcal{Y}s}) = (\widetilde{q \circ m}, \widetilde{q_2 \circ m})$$

Because of the commutativity of the original diagram, $\widetilde{q \circ m} = \tilde{p_1}$; because q_2 has been obtained from the path bisimulation condition, $\widetilde{q_2 \circ m} = \tilde{p_2}$. That is, $R(s)(\tilde{k}) = (\tilde{p_1}, \tilde{p_2}) = \tilde{p}$. Since that is also equal to $\widetilde{k \circ m}$, we obtain the commutativity of the upper triangle

$$k \circ m = p$$

This concludes the proof of the $\mathcal{Y}\mathbf{P}$-openness of π_1. The fact that π_2 is $\mathcal{Y}\mathbf{P}$-open can be shown similarly. This means that X_1 and X_2 are connected by a span of open morphisms π_1, π_2, which means that they are $\mathcal{Y}\mathbf{P}$-bisimilar.

□

In specific cases of synchronization trees and event structures, strong path bisimulation also coincide with $\mathbf{P}$-bisimulation.

Theorem 14. *1. Two synchronization trees X_1 and X_2 are strong path bisimilar iff they are $\mathbf{Bran}_L$-bisimilar.*

2. Two event structures X_1 and X_2 are strong path bisimilar iff they are $\mathbf{Pom}_L$-bisimilar.

The finishing touch that we need to prove theorem 10 is that strong path bisimulation is preserved under the canonical functor.

Lemma 1. *Let* $\mathbf{P}$ *be a full, dense subcategory of* $\mathbf{M}$*. Two objects* X_1, X_2 *of* $\mathbf{M}$ *are strongly path bisimilar iff* $C(X_1), C(X_2)$ *are strongly path bisimilar in* $[\mathbf{P}^{op}, \mathbf{Set}]_r$*.*

Proof. $\Rightarrow$

Let R_0 be a strong path bisimulation relation between objects X_1 and X_2 of $\mathbf{M}$. Take R to be

$$R = \{(Cp_1, Cp_2) \mid p_1 : P \to X_1, p_2 : P \to X_2, (p_1, p_2) \in R_0\}$$

We claim that R is a strong path bisimulation between $C(X_1)$ and $C(X_2)$. First of all, because unique morphisms $i_1 : I \to X_1$ and $i_2 : I \to X_2$ are related by R_0, their images under C are related by R. Therefore, the unique arrows from the initial object $\mathcal{Y}(I) = C(I)$ in the category of rooted presheaf to X_1 and X_2 are related by R, satisfying the first condition of path bisimulation. Secondly, we note that because the inclusion $\mathbf{P} \hookrightarrow \mathbf{M}$ is dense, the canonical functor C is full and faithful; furthermore, C is equivalent to $\mathcal{Y}$, when restricted to $\mathbf{P}$. Now, consider the second condition of path bisimulation:

$$\begin{tikzcd} & CP \arrow[dl, "Cp_1"'] \arrow[d, "q"] \arrow[dr, "Cp_2"] & \\ CX_1 & CQ \arrow[l, "m"] & CX_2 \end{tikzcd}$$

In this situation, $Cp_1 = m \circ q$. Because C is full, $m = Cm'$ and $q = Cq'$ for some morphisms m', q' in $\mathbf{M}$. By the faithfullness, $p_1 = m' \circ q'$. By the strong path bisimulation R_0 there exists a morphism $k : Q \to X_2$, such that $(m', k) \in R_0$ and $k \circ q' = p_2$. By the definition of R we have that $(Cm' = m, Ck) \in R$, and the following diagram commutes:

$$\begin{tikzcd} & CP \arrow[dl, "Cp_1"'] \arrow[d, "q"] \arrow[dr, "Cp_2"] & \\ CX_1 & CQ \arrow[l, "m"] \arrow[r, "Ck"'] & CX_2 \end{tikzcd}$$

which is exactly what we need for the second condition of path bisimulation to be satisfied. The other condition can be verified in a similar fashion. The strongness condition can be checked similarly as well.

$\Leftarrow$

Assume that R_0 is the strong path bisimulation between $C(X_1)$ and $C(X_2)$. Define R to be

$$R = \{(p_1, p_2) \mid (C(p_1), C(p_2)) \in R_0\}$$

We have to verify that R is a strong path bisimulation between X_1 and X_2.

Clearly, if i_1, i_2 are the unique morphisms from I to X_1, X_2, then $C(i_1)$ and $C(i_2)$ are the unique morphisms from the initial rooted presheaf to $C(X_1), C(X_2)$. Therefore, $(i_1, i_2) \in R$.

Assume that the following diagram commutes:

$$
\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[d, "q"] \arrow[dr, "p_2"] & \\
X_1 & Q \arrow[l, "m"] & X_2
\end{tikzcd}
$$

with $(p_1, p_2) \in R$. Keeping in mind the fact that $\mathcal{Y}$ and C are equivalent on $\mathbf{P}$, we look at the image of this diagram under C.

$$
\begin{tikzcd}
 & CP \arrow[dl, "Cp_1"'] \arrow[d, "Cq"] \arrow[dr, "Cp_2"] & \\
CX_1 & CQ \arrow[l, "Cm"] & CX_2
\end{tikzcd}
$$

By the functor laws, the diagram commutes. Furthermore, by definition of R, $(C(p_1), C(p_2)) \in R_0$. Because R_0 is a strong path bisimulation, there is a morphism $k : CQ \to CX_2$, such that $(Cm, k) \in R_0$ and $k \circ Cq = Cp_2$. Because C is full, $k = Ck'$ for some k'.

$$
\begin{tikzcd}
 & CP \arrow[dl, "Cp_1"'] \arrow[d, "Cq"] \arrow[dr, "Cp_2"] & \\
CX_1 & CQ \arrow[l, "Cm"] \arrow[r, "Ck'"'] & CX_2
\end{tikzcd}
$$

By definition of R, $(m, k') \in R$. Using functorial laws, and the fact that C is faithful, we "go back" from $[\mathbf{P}^{op}, \mathbf{Set}]_r$ to $\mathbf{M}$, in which the following diagram commutes:

$$
\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[d, "q"] \arrow[dr, "p_2"] & \\
X_1 & Q \arrow[l, "m"] \arrow[r, "k'"'] & X_2
\end{tikzcd}
$$

This ensures that the second condition of strong path bisimulation is satisfied. The symmetric condition can be verified similarly. The strongness condition can be checked easily as well.

□

Now the proof of theorem 10 becomes easy.

Proof (of Theorem 10). Let S_1, S_2 be two synchronization trees. By theorem 14, we know that S_1 and S_2 are $\mathbf{Bran}_L$-bisimilar iff they are strong path-bisimilar. By lemma 1, they are strong path-bisimilar iff their images $C(S_1)$ and $C(S_2)$ are strong path-bisimilar. By theorem 13, $C(S_1)$ and $C(S_2)$ are strong path-bisimilar iff they are $C(\mathbf{Bran}_L)$-bisimilar (or, equivalently, $\mathcal{Y}\mathbf{Bran}_L$-bisimilar).

The proof of the second clause is practically the same. □

4.4 Presheaves as transition systems

The familiar construction of the category of elements, with slight modifications, can be used to obtain a transition system of a presheaf $X : \mathbf{P}^{op} \to \mathbf{Set}$. This allows us to view presheaves as transition systems.

Definition 10. *A transition system* $\mathrm{e}l(X)$ *for a rooted presheaf* X *is a defined as following:*

- *The states of the transition system are of form* (x, P)*, where* P *is an object of* $\mathbf{P}$*, and* $x \in X(P)$*.*
- *There is a transition* $(x, P) \xrightarrow{m} (y, Q)$ *if* $m : P \to Q$ *is a morphism of* $\mathbf{P}$*, and* $X(m)(y) = x$*.*
- *The initial state is* (i, I)*, where* I *is the initial object of* $\mathbf{P}$ *and* i *is the unique member of* $X(I)$*.*

In fact, this construction $\mathrm{e}l$ can be seen as a functor $\mathrm{e}l : [\mathbf{P}^{op}, \mathbf{Set}]_r \to \mathbf{T}$, which acts on morphisms $f : X \Rightarrow Y$ as following:

$$\mathrm{e}l(f) : \mathrm{e}l(X) \to \mathrm{e}l(Y)$$
$$\mathrm{e}l(f)(P, p) = (P, f_P(p))$$

It is easy to see that $\mathrm{e}l(f)$ is a morphism of transition systems. Let $(P, p) \xrightarrow{m} (Q, q)$ be a step in $\mathrm{e}l(X)$. That means that $X(m)(q) = p$. From commutativity of the following diagram

$$\begin{tikzcd} X(P) & X(Q) \arrow[l, "X(m)"'] \arrow[d, "f_Q"] \\ Y(P) \arrow[from=u, "f_P"'] & Y(Q) \arrow[l, "Y(m)"] \end{tikzcd}$$

we obtain that $Y(m)(f_Q(q)) = f_P(p)$, i.e. $(P, f_P(p)) \xrightarrow{m} (Q, f_Q(q))$ in Y. Does $\mathrm{e}l(f)$ preserve initial states? Well, $\mathrm{e}l(f)(I, i_0^X) = (I, f_I(i_0^X))$. But $f_I(i_0^X)$ is exactly i_0^Y (what else could it be?).

In fact, we can show that $\mathrm{e}l : [\mathbf{P}^{op}, \mathbf{Set}]_r \to \mathbf{T}$ is a full functor.

Proposition 6. *If* $f : \mathrm{e}l(X) \to \mathrm{e}l(Y)$ *is a morphism of transition systems, then there is a natural transformation* $g : X \Rightarrow Y$*, such that* $\mathrm{e}l(g) = f$*.*

Proof. Put $g_P(p) = \pi_2(f(P, p))$. We need to ensure the naturality of g. Consider the following "square"

$$\begin{tikzcd} X(P) & X(Q) \arrow[l, "X(m)"'] \arrow[d, "g_Q"] \\ Y(P) \arrow[from=u, "g_P"'] & Y(Q) \arrow[l, "Y(m)"] \end{tikzcd}$$

for a morphism $m : P \to Q$. Let q be an element of $X(Q)$. By "going" left-down from $X(Q)$ we get $g_P(X(m)(q)) = \pi_2(f(P, X(m)(q)))$. By "going" down-left we obtain $Y(m)(g_Q(q)) = Y(m)(\pi_2(f(Q, q)))$.

It follows trivially from the definition of $\mathrm{e}l$, that there is a transition $(P, X(m)(q)) \xrightarrow{m} (Q, q)$ in $\mathrm{e}l(X)$. Because f is a morphism of transition systems, there is a transition $f(P, X(m)(q)) \xrightarrow{m} f(Q, q)$ in $\mathrm{e}l(Y)$. That is, $Y(m)(\pi_2(f(Q, q))) = \pi_2(f(P, X(m)(q)))$. This ensures the naturality of g. □

The fact that el is full provides us with some interesting results. We can view $[\mathbf{P}^{op}, \mathbf{Set}]_r$ as a full subcategory of $\mathbf{T}$. Then, theorem 6 tells us that a morphism is $\mathcal{Y}\mathbf{P}$-open in $[\mathbf{P}^{op}, \mathbf{Set}]_r$ iff it is $el(\mathcal{Y}\mathbf{P})$-open in $\mathbf{T}$. We have a nice alternative way of computing whether a morphism is open.

Do we have any results akin to that about bisimulation? Well, as it turns out, $\mathbf{P}$-bisimulation in the category of rooted presheaves corresponds to *back-and-forth bisimulation* between transition systems.

Definition 11. *A relation R between two transition systems T_1 and T_2 is called a* back-and-forth bisimulation *if*

- *R is a bisimulation;*
- *If $s_1 R s_2$ and $T_1 : s_1' \xrightarrow{a} s_1$, then there is a state s_2', such that $T_2 : s_2' \xrightarrow{a} s_2$ and $s_1' R s_2'$.*
- *If $s_1 R s_2$ and $T_2 : s_2' \xrightarrow{a} s_2$, then there is a state s_1', such that $T_1 : s_1' \xrightarrow{a} s_1$ and $s_1' R s_2'$.*

Theorem 15. *Let X_1, X_2 be rooted presheaves. X_1 and X_2 are $\mathbf{P}$-bisimilar iff* $\mathrm{el}(X_1)$ *and* $\mathrm{el}(X_2)$ *are back-and-forth bisimilar.*

Proof. By theorem 13, $\mathbf{P}$-bisimulation between rooted presheaves is equivalent to strong path bisimulation. Therefore, it suffices to verify that X_1 and X_2 are strong path bisimilar iff $el(X_1)$ and $el(X_2)$ are back-and-forth-bisimilar.

$\Rightarrow$ Let R_0 be a strong path bisimulation relation between X_1 and X_2. Take

$$R = \{((P, p), (P, p')) \mid (\tilde{p}, \tilde{p'}) \in R_0\}$$

We now verify that R is a back-and-forth bisimulation.

1. $\mathrm{el}(X_1) : (P, p) \xrightarrow{m} (Q, q)$ and $(P, p)R(P, p')$. Then, by definition, $X_1(m)(q) = p$. By the Yoneda lemma, this means that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{Y}P \arrow[dl, "\tilde{p}"'] \arrow[d, "\mathcal{Y}m"] \\ X_1 & \mathcal{Y}Q \arrow[l, "\tilde{q}"] \end{tikzcd}$$

   Because $\tilde{p} R_0 \tilde{p'}$, there exists a map $\tilde{q'} : \mathcal{Y}Q \to X_2$, making the following diagram commute

$$\begin{tikzcd} & \mathcal{Y}P \arrow[dl, "\tilde{p}"'] \arrow[d, "\mathcal{Y}m"] \arrow[dr, "\tilde{p'}"] & \\ X_1 & \mathcal{Y}Q \arrow[l, "\tilde{q}"] \arrow[r, "\tilde{q'}"'] & X_2 \end{tikzcd}$$

   Which means that $\tilde{q'} \circ \mathcal{Y}m = \tilde{p'}$, or, in terms of the Yoneda and presheaves, $X_2(m)(q') = p'$. That means that there is a step $(P, p') \xrightarrow{m} (Q, q')$, and $(P, p')R(Q, q')$.

2. The symmetric condition can be proved in the same manner.
3. Assume that $\mathrm{el}(X_1) : (P, p) \xrightarrow{m} (Q, q)$ and $(Q, q)R(Q, q')$. That means that $X(m)(q) = p$. Via Yoneda lemma we obtain a commutative diagram

$$\begin{tikzcd} & \mathcal{Y}P \arrow[dl, "\tilde{p}"'] \arrow[d, "\mathcal{Y}m"] & \\ X_1 & \mathcal{Y}Q \arrow[l, "\tilde{q}"] \arrow[r, "\tilde{q'}"'] & X_2 \end{tikzcd}$$

with $\tilde{q}R_0\tilde{q'}$. Because R_0 is a *strong* path bisimulation, we know that $\tilde{p} = (\tilde{q} \circ \mathcal{Y}m)R_0(\tilde{q'} \circ \mathcal{Y}m)$. This immediately gives us that

$\mathrm{el}(X_2) : (P, p') \xrightarrow{m} (Q, q')$, and $(P, p)R(P, p')$, where $\tilde{p'} = \tilde{q'} \circ \mathcal{Y}m$.
4. It is clear that the initial states are related.

□

5 Going further: relational presheaves

The approach that was considered so far in this document relies on the assumption that the set of labels is a "pure set" without any structure. A generalization, presented by P. Sobociński in [5], tries to deal with this problem by viewing transition systems as *relational presheaves.*

A relational presheaf is a lax contravariant functor (the precise meaning of this term will be explained below) from a category $\mathscr{C}$ to the category of sets with relations **Rel**. We will write $[\mathscr{C}^{op}, \mathbf{Rel}]$ for the category of relational presheaves.

Sobocinski explains the idea by starting from a more familiar coalgebraic approach to concurrency.

Definition 12. *Let $F : \mathscr{C} \to \mathscr{C}$ be an endofunctor. A* coalgebra *for F is a morphism $\theta : X \to F(X)$ for some object X of $\mathscr{C}$.*

In particular, we can consider an endofunctor $\mathcal{P}(A \times -) : \mathbf{Set} \to \mathbf{Set}$, which sends a set X to a powerset over $A \times X$. Then, every coalgebra $f : X \to \mathcal{P}(A \times X)$ over such endofunctor can be viewed as a transition system without an initial state. There is a transition $s \xrightarrow{a} s'$ between two states $s, s' \in X$ if $(a, s') \in f(s)$.

Conversely, every transition system without an initial state gives rise to a powerset coalgebra.

If one wants to consider additional structure on the set of labels A, then one should pull out the set A from being under $\mathcal{P}$. This is achieved by the following series of derivations:

$$f : X \to \mathcal{P}(A \times X)$$

is equivalent to

$$f' : X \to \mathcal{P}(X)^A$$

which is in turn equivalent to

$$f'' : A \to \mathcal{P}(X)^X$$

One can also note that the exponent object $\mathcal{P}(X)^X$ is equivalent to a morphism $X \to X$ in a Kleisli category of the $\mathcal{P}$ monad

$$f''' : A \to Hom_{Kl(\mathcal{P})}(X, X)$$

We can decide to equip A with additional structure, for example, the structure of a monoid. Then A can be seen as a single-object category $\mathbf{C}$, and f''' can be seen as a functor

$$F : \mathbf{C} \to Kl(\mathcal{P})$$

Finally, we can notice that $Kl(\mathcal{P})$ is just **Rel** – the category of sets with relations.

This brings us to the definition that is presented in [5].

Definition 13. *A relational presheaf is a lax functor* $F : \mathbf{C}^{op} \to \mathbf{Rel}$.

The laxness in the definition means that $id_{F(X)} \subseteq F(id_X)$ and $F(B) \circ F(A) \subseteq F(A \circ B)$.

Morphisms of relational presheaves are *oplax natural transformations*, i.e. $\phi : F \Rightarrow F'$ is a morphism of relational presheaves if $F(f) \circ \phi_D \subseteq F'(f) \circ \phi_C$ for all objects C, D and morphisms $f : C \to D$.

The author of [5] then goes on about showing that labeled transition systems with the monoidal structure M on labels can be represented by relational presheaves $M \to \mathbf{Rel}$, and morphisms of such presheaves are exactly simulations between transition systems.

Ordinary labeled transition system (without an initial state) over a labeling set A can be represented as relational presheaves $(A^*)^{op} \to \mathbf{Rel}$, where A^* is the category of strings over A. The morphisms of such presheaves are simulations as well. In this case, the representation of ordinary LTSs are very similar to the representation using regular presheaves.

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